

The University of Chicago

SYSTEMS OF QUADRICS ASSOCIATED
WITH A POINT OF A SURFACE

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1937

By
LOUIS GREEN

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THE UNIVERSITY OF CHICAGO

SYSTEMS OF ORNAMENTAL DESIGN
WITH A POINT OF VIEW

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SYSTEMS OF QUADRICS ASSOCIATED WITH A POINT OF A SURFACE.*

By LOUIS GREEN.

1. Introduction. One of the popular methods of studying the local geometry of an analytic surface immersed in a three-dimensional projective space is to examine the properties of algebraic osculants. For, these osculants can be used to obtain simple characterizations of many elements intimately related to the given surface. The osculating quadrics at a point of a surface are of particular importance in this respect, and it is the purpose of this paper to make a more comprehensive study of certain of these quadrics, obtaining as a result significant properties of the surface.

Among other things, the problem of determining the envelope of a two-parameter family of Moutard quadrics is solved, the singularity of the curve of intersection of the surface and one of its Moutard quadrics is investigated, certain individual members of each Moutard pencil of quadrics¹ are characterized, and three new pencils of quadrics which we call the Darboux-Segre pencils of quadrics are defined and discussed.

2. Analytic basis. The local coördinate tetrahedron of G. M. Green (with a modified unit point) is taken as the projective coördinate system throughout this paper, and the equations of the various configurations to be used are obtained in this section by transformation from Fubini's coördinate system.

Fubini's canonical differential equations of an analytic curved surface S are given by

$$x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v, \quad (\theta = \log \beta \gamma).$$

Let O be an ordinary point of S , chosen as the vertex $(1, 0, 0, 0)$ of two local tetrahedrons, one of Fubini and the other of Green. If a point of the ambient three-space has the two sets of coördinates $(x_1, \dots, x_4)$, $(X_1, \dots, X_4)$ relative to these respective tetrahedrons, then the following relations exist:²

* Presented to the Society, April 9, 1937. Received by the Editors October 30, 1937.

¹ So called because each of these pencils contains a quadric of Moutard. These pencils were first studied by Su and Ichida, *Japanese Journal of Mathematics*, vol. 10 (1933), pp. 209-216.

² Lane, "Power series expansions in the neighborhood of a point on a surface," *Proceedings of the National Academy of Sciences*, vol. 13 (1927), pp. 808-813.

$$\begin{aligned}
 (1) \quad & x_1 = 4X_1 + 3l\phi X_2 + 3m\psi X_3 + (9/4)n(\phi\psi - 8\theta_{uv})X_4, \\
 & x_2 = -12lX_2 - 9n\psi X_4, \\
 & x_3 = -12mX_3 - 9n\phi X_4, \\
 & x_4 = 36nX_4,
 \end{aligned}$$

where l, m, n, ϕ, ψ , have the values

$$\begin{aligned}
 l &= 1/(\beta^2\gamma)^{1/3}, & m &= 1/(\beta\gamma^2)^{1/3}, & n &= lm, \\
 \phi &= (\log \beta\gamma^2)_u, & \psi &= (\log \beta^2\gamma)_v.
 \end{aligned}$$

With the introduction of non-homogeneous coördinates X, Y, Z defined in the customary way, 0 becomes the origin of a local non-homogeneous coördinate system, and the coördinates of a point on S sufficiently "near" 0 satisfy a convergent power series expansion of the form

$$\begin{aligned}
 (2) \quad Z &= XY + X^3 + Y^3 + AX^3Y + BXY^3 + C_0X^5 + C_1X^4Y \\
 &\quad + C_2X^3Y^2 + C_3X^2Y^3 + C_4XY^4 + C_5Y^5 + \dots,
 \end{aligned}$$

where all the coefficients are absolute invariants of the surface, and in particular,

$$\begin{aligned}
 A &= -3m\psi/2, & B &= -3l\phi/2, \\
 20C_0 &= 36l^2p - 9l^2\phi_u - 27m\psi - (9/4)l^2\phi^2 + 9l^2\phi\theta_u, \\
 16C_1 &= 24n\psi_u - 12n\phi_v + 135n\phi\psi - 135, \\
 (3) \quad 16C_2 &= -135m^2q + 72m^2\psi_v + 27m^2\psi^2 - 180l\phi + 72m^2\psi(\beta_v/\beta), \\
 16C_3 &= -135l^2p + 72l^2\phi_u + 27l^2\phi^2 - 180m\psi + 72l^2\phi(\gamma_u/\gamma), \\
 16C_4 &= 24n\phi_v - 12n\psi_u + 135n\phi\psi - 135, \\
 20C_5 &= 36m^2q - 9m^2\psi_v - 27l\phi - (9/4)m^2\psi^2 + 9m^2\psi\theta_v.
 \end{aligned}$$

For use in § 7 the formulas for the derivatives of the local coördinates $X_1, \dots, X_4$ will now be obtained. Equations (1) are first solved for the X_i , the resulting equations are then differentiated with respect to $u(v)$, and the x_{iu} (x_{iv}) which appear are replaced by their known values³ in terms of the x_i . The right members are then expressed in terms of the X_i by means of (1), and when simplified with the use of (3) take the form

$$\begin{aligned}
 36lX_{1u} &= 6BX_1 + (60C_0 - 72A)X_2 + (12C_4 - 45AB + 108)X_3 \\
 &\quad + (36C_2 - 108B - 54A^2)X_4, \\
 36lX_{2u} &= 12X_1 + 2BX_2 + (12C_4 - 45AB + 216)X_4, \\
 36lX_{3u} &= -36X_2 - 2BX_3 + (60C_0 - 108A)X_4, \\
 36lX_{4u} &= 12X_3 - 6BX_4; \\
 (4) \quad 36mX_{1v} &= 6AX_1 + (12C_1 - 45AB + 108)X_2 + (60C_5 - 72B)X_3 \\
 &\quad + (36C_3 - 108A - 54B^2)X_4, \\
 36mX_{2v} &= -2AX_2 - 36X_3 + (60C_5 - 108B)X_4, \\
 36mX_{3v} &= 12X_1 + 2AX_3 + (12C_1 - 45AB + 216)X_4, \\
 36mX_{4v} &= 12X_2 - 6AX_4.
 \end{aligned}$$

³ Lane, *Projective Differential Geometry of Curves and Surfaces*, Chicago, 1932, p. 111.

By means of transformation (1) all the remaining results of this section are readily established.

The tangents of Darboux through 0 are given by

$$X_2^3 + X_3^3 = 0 = X_4;$$

the equation of a quadric of Darboux,

$$x_2x_3 - x_1x_4 + hx_4^2 = 0 \quad (h = h(u, v)),$$

becomes

$$(5) \quad X_2X_3 - X_1X_4 + KX_4^2 = 0, \quad K = (9/2\beta\gamma)(\theta_{uv} + 2h);$$

the equations of a canonical line of the second kind,

$$x_1 - k\phi x_2 - k\psi x_3 = 0 = x_4,$$

have the form

$$X_1 - (B/2)(1 + 4k)X_2 - (A/2)(1 + 4k)X_3 = 0 = X_4,$$

and its polar line with respect to a quadric of Darboux joins 0 to the point

$$(0, (A/2)(1 + 4k), (B/2)(1 + 4k), 1).$$

If the corresponding equations of a plane in the above-mentioned coördinate systems are

$$\sum_1^4 \xi_i x_i = 0, \quad \sum_1^4 \eta_i X_i = 0,$$

then the transformation of Čech in Fubini's system,

$$\begin{aligned} \rho x_1 &= \xi_2 \xi_3 \xi_4 - j(\gamma \xi_2^3 + \beta \xi_3^3), & \rho x_2 &= -\xi_2 \xi_3^2, \\ \rho x_3 &= -\xi_2^2 \xi_3, & x_4 &= 0 \end{aligned} \quad (j = \text{const.}),$$

is expressed in Green's system by the equations

$$\begin{aligned} \sigma X_1 &= \eta_2 \eta_3 \eta_4 + 3j(\eta_2^3 + \eta_3^3), & \sigma X_2 &= -\eta_2 \eta_3^2, \\ \sigma X_3 &= -\eta_2^2 \eta_3, & X_4 &= 0. \end{aligned}$$

A curve on S which passes through 0 in a non-asymptotic direction and satisfies the differential equation

$$dv - \lambda du = 0 \quad (\lambda = \lambda(u, v)),$$

has for its osculating plane at 0 the plane represented by

$$2\lambda(\lambda x_2 - x_3) + (\beta - \gamma\lambda^3 - \theta_u\lambda + \theta_v\lambda^2 + \lambda')x_4 = 0 \quad (\lambda' = v'' = \lambda_u + \lambda\lambda_v).$$

Under transformation of coördinates (1) this equation becomes

$$(6) \quad 6\mu(\mu X_2 - X_3) + (9\mu^3 - 9 + A\mu^2 - B\mu - 9\mu')X_4 = 0,$$

where μ, μ' , are defined by

$$(7) \quad \mu = \lambda l/m, \quad \mu' = l(\mu_u + \lambda \mu_v) = l\mu_u + m\mu_v.$$

The osculating plane at 0 of the pangeodesic which passes through this point in a non-Darboux direction $X_3/X_2 = \mu$ is expressed by

$$(8) \quad 2\mu(\mu^3 + 1)(\mu X_2 - X_3) + (3\mu^6 - 3 + A\mu^2 - B\mu^4)X_4 = 0,$$

and the quadric of Moutard in the non-asymptotic direction μ at 0 has the equation

$$(9) \quad \mu^3(X_2X_3 - X_1X_4) + \mu^2(2 - \mu^3)X_2X_4 + \mu(2\mu^3 - 1)X_3X_4 \\ + (A\mu^2 + B\mu^4 - \mu^6 - 2\mu^3 - 1)X_4^2 = 0.$$

3. The curve of intersection of a surface and a quadric. The equation of the most general quadric having second-order contact with the surface S at the point 0 is

$$(10) \quad XY - Z + K_2XZ + K_3YZ + K_4Z^2 = 0,$$

where K_2, K_3, K_4 are arbitrary functions of u and v . The curve of intersection C of this quadric and S is expressed by (10) and

$$(11) \quad 0 = X^3 - K_2X^2Y - K_3XY^2 + Y^3 + (A - K_2^2)X^3Y \\ - (2K_2K_3 + K_4)X^2Y^2 + (B - K_3^2)XY^3 + \dots,$$

the latter equation being obtained by solving (10) for Z as a power series in X, Y and then subtracting from (2). The curve C has a triple-point at 0 with triple-point tangents satisfying the equations

$$(12) \quad X^3 - K_2X^2Y - K_3XY^2 + Y^3 = 0 = Z.$$

Comparison of equations (10) and (12) leads to the following conclusion:

Two arbitrary non-asymptotic tangents to S at 0 determine a pencil of quadrics having second-order contact with S at 0 and meeting S in a one-parameter family of curves each containing these tangents as two of its triple-point tangents. The residual tangent at 0 is the same for all the curves of the family.

A particular quadric of such a pencil is uniquely determined, ordinarily, by characterizing the osculating plane at 0 of *any one* of the three branches of the curve C .⁴

⁴ The Moutard pencils of quadrics are the only pencils for which this is not the case. But, as is shown in §§ 5, 6, for such a pencil of quadrics in a direction μ the

Several particular examples of the above theorem are worth noting. If two of the triple-point tangents lie in distinct Darboux directions at 0, the third tangent must lie in the residual Darboux direction and the resulting quadrics are the quadrics of Darboux.

If two of the tangents coincide in a non-asymptotic direction $X_3/X_2 = \mu$, the third must lie in the direction $-1/\mu^2$, and the quadrics determined form the Moutard pencil of quadrics in the direction μ . If $\mu^3 = -1$ the triple-point tangents all coincide in one of the Darboux directions, while if $\mu^3 = 1$ two of the tangents coincide in a Segre direction and the third tangent lies in the conjugate Darboux direction.

If two of the tangents lie in distinct Segre directions, the third is seen to lie in that Darboux direction which is conjugate to the residual Segre direction. Every Darboux direction thus determines a different pencil of such quadrics, so that each of the three pencils may be designated as the Darboux-Segre pencil of quadrics in the particular Darboux direction considered.

The Darboux quadrics and the three Darboux-Segre pencils of quadrics are special cases of the pencil

$$XY - Z - \frac{\mu^3 + 1}{\mu} XZ - \frac{\mu^3 + 1}{\mu^2} YZ + K_4 Z^2 = 0 \quad (K_4 \text{ arbitrary}),$$

which is determined by the tangents in the directions $\mu\omega, \mu\omega^2, -1/\mu^2$, where ω is a primitive cube root of unity.

The quadrics of Darboux are thus seen to constitute the only pencil which can be determined by three tangents apolar to the asymptotic tangents at 0. Similarly, there exists a unique triple of tangents apolar to a given pair of conjugate tangents $X_3/X_2 = \pm \mu, X_4 = 0$, which according to the above theorem can determine a pencil of quadrics. These quadrics, whose equations can be shown to have the form

$$XY - Z - 3\mu^2 XZ - (3/\mu^2) YZ + K_4 Z^2 = 0, \quad (K_4 \text{ arbitrary}),$$

were defined by Davis⁵ in an entirely different manner.

4. Quadrics associated with the tangents of Darboux and of Segre.

The theorem of the last section is applied here to the quadrics of Darboux, the Moutard pencils in the Segre directions, and the Darboux-Segre pencils of quadrics.

osculating plane at 0 of a properly chosen branch of C does determine a unique quadric of the pencil, except when μ is a Darboux direction. In this latter case the osculating plane at 0 of each of the three branches of C coincides with the tangent plane to S , unless the quadric is the quadric of Moutard of the pencil.

⁵ "Contributions to the theory of conjugate nets," *Chicago Dissertation* (1933), p. 12.

The surface S and an arbitrary quadric of Darboux with parameter K meet in a curve C whose three branches are representable, as is seen from (2) and (5), by the series

$$\begin{aligned} Y &= -\epsilon_i X + \frac{1}{3}(A\epsilon_i^2 + B\epsilon_i + K)X^2 + mX^3 + \cdots, \\ Z &= -\epsilon_i X^2 + \frac{1}{3}(A\epsilon_i^2 + B\epsilon_i + K)X^3 + \cdots, \end{aligned} \quad (i = 1, 2, 3),$$

where

$$(13) \quad -9\epsilon_i^2 m = 2AB\epsilon_i + 2B^2 + AK + 3BK\epsilon_i^2 + K^2\epsilon_i + 3 \sum_{j=0}^5 C_j (-\epsilon_i)^j,$$

$$\epsilon_1 = 1, \quad \epsilon_2 = \omega, \quad \epsilon_3 = \omega^2$$

and ω is a primitive cube root of unity. The osculating plane at 0 of each branch of C has the equation

$$(14) \quad \epsilon_i^2 X + \epsilon_i Y + \frac{1}{3}(A\epsilon_i^2 + B\epsilon_i + K)Z = 0,$$

from which it follows that *these osculating planes are coaxial if, and only if, the given quadric is the quadric of Wilczynski ($K = 0$). The common axis of the three planes then becomes the canonical line of the first kind for which $k = -5/12$, which we shall designate as the first axis of Bompiani.*⁶

It may be remarked here that the Moutard quadric in a Darboux direction $-\epsilon_i$ meets a Darboux quadric (5) in the asymptotic tangents through 0 and in a conic lying in the plane (14). As a consequence the Moutard quadrics in the three Darboux directions have a residual point of intersection on the first axis of Bompiani.

If the parameter K is left arbitrary, then the planes (14) correspond under a transformation of Čech with parameter j to three collinear points if, and only if, $K = -18j$ (the line thereby determined being the polar with respect to a Darboux quadric of the first axis of Bompiani, independently of the value of K). This equation defines a correspondence at each point of the surface between the Čech transformations and those Darboux quadrics for which K is independent of u, v . Thus, the quadrics of Lie and Fubini for which $K = -9/2$ and -3 respectively, characterize the transformations of Čech for which $j = 1/4$ and $1/6$, whereas the transformation of Segre ($j = 1$) characterizes the quadric of Darboux for which $K = -18$.

The osculating plane (14) is a stationary osculating plane if, and only if,

$$-9\epsilon_i m = (A\epsilon_i^2 + B\epsilon_i + K)^2;$$

the value of K is thereby uniquely determined⁷ and when substituted into

⁶ This line was first characterized in a more complicated way by Bompiani in the *Rendiconti dei Lincei*, ser. 5, vol. 33₁ (1924), pp. 85-90.

⁷ The value of K depends on the particular Darboux direction chosen, so that there exist three distinct Darboux quadrics with the property considered.

(14) gives as the equation of the stationary osculating plane

$$(15) \quad (A\epsilon_i - B)(\epsilon_i X + Y) + \left[\sum_{j=0}^5 C_j (-\epsilon_i)^j \right] Z = 0.$$

The Moutard pencil of quadrics at 0 in the Segre direction $X_3/X_2 = \epsilon_i$ has the form

$$(16) \quad XY - Z + \epsilon_i^2 XZ + \epsilon_i YZ + K_i Z^2 = 0,$$

with K_i as parameter of the pencil. The curve of intersection of the surface S and an arbitrary quadric of this pencil has a branch in the conjugate Darboux direction, the osculating plane at 0 of this branch being expressed by

$$(17) \quad \epsilon_i^2 X + \epsilon_i Y + \frac{1}{4}(A\epsilon_i^2 + B\epsilon_i + K_i)Z = 0.$$

Since the equation of the osculating plane at 0 of the curve of Darboux in the direction $-\epsilon_i$ is

$$(18) \quad \epsilon_i^2 X + \epsilon_i Y + \frac{1}{6}(A\epsilon_i^2 + B\epsilon_i - 18)Z = 0,$$

it follows that the planes (17) and (18) coincide only if

$$K_i = -(A/3)\epsilon_i^2 - (B/3)\epsilon_i - 12.$$

This value of K_i determines a unique quadric of the pencil (16), and since there are three Moutard pencils in the Segre directions at 0, three corresponding quadrics are obtained in all. *These quadrics intersect in the asymptotic tangents through 0 and in a residual point lying on the canonical line of the first kind for which $k = -1/12$. Under the transformation of Čech with $j = 1/2$ the three planes (18) are carried into collinear points on the second axis of Čech.*

The condition that the osculating plane (17) contain a line l joining 0 to an arbitrary point $(0, a, b, 1)$ requires that

$$K_i = -(4a + A)\epsilon_i^2 - (4b + B)\epsilon_i.$$

The line l obtains in this way a single quadric out of each of the three Moutard pencils in the Segre directions, the residual point of intersection P of these quadrics having the coördinates

$$(4a + A)(4b + B), \quad 4a + A, \quad 4b + B, \quad 1.$$

As the line l varies in the bundle at 0, the point P generates the quadric of Wilczynski; P lies on l if, and only if, l is the first axis of Bompiani.

The quadric of Moutard in the Segre direction ϵ_i at 0 is that quadric of the pencil (16) for which

$$K_i = A\epsilon_i^2 + B\epsilon_i - 4;$$

the osculating plane (17) determined by this value of K_i can also be described as the plane containing the non-degenerate conic of intersection of the Moutard quadrics in the conjugate directions $\pm \epsilon_i$. The residual point of intersection of the Moutard quadrics in the three Segre directions lies on the canonical line of the first kind for which $k = -3/4$, while the transformation of Čech for which $j = 1/6$ carries the three planes (17) determined by these quadrics into collinear points on the second directrix of Wilczynski.

The Darboux-Segre pencils of quadrics will now be considered. For each choice of ϵ_i let $\epsilon_i, \epsilon_j, \epsilon_k$ represent the three values in (13). Then the three directions

$$X_3/X_2 = -\epsilon_i, \quad X_3/X_2 = \epsilon_j, \quad X_3/X_2 = \epsilon_k$$

at 0 determine a Darboux-Segre pencil of quadrics in the Darboux direction $-\epsilon_i$, having the equation

$$(19) \quad XY - Z - 2\epsilon_i^2 XZ - 2\epsilon_i YZ + K_i Z^2 = 0 \quad (K_i \text{ arbitrary}).$$

If the osculating planes at 0 of the three branches of the curve of intersection of the surface S and one of these quadrics be denoted by $\pi_{ii}, \pi_{ij}, \pi_{ik}$ corresponding to the respective directions $-\epsilon_i, \epsilon_j, \epsilon_k$, then

$$\pi_{ii}: \quad \epsilon_i^2 X + \epsilon_i Y + (A\epsilon_i^2 + B\epsilon_i + K_i)Z = 0,$$

$$\pi_{ih}: \quad \epsilon_h^2 X - \epsilon_h Y + P_{ih}Z = 0 \quad (h = j, k),$$

where

$$P_{ih} = \frac{-A\epsilon_h - B + 4\epsilon_h^2 + K_i\epsilon_h^2}{2\epsilon_i^2 + 4\epsilon_i\epsilon_h + 3\epsilon_h^2}.$$

The condition that the plane π_{ii} coincide with the osculating plane of a curve of Darboux at 0 determines a unique quadric out of each Darboux-Segre pencil. *The residual point of intersection of these three quadrics is the point of intersection (besides 0) of the quadric of Fubini with the canonical line of the first kind for which $k = -11/24$.*

A unique quadric in each Darboux-Segre pencil is also obtained from the condition that the three planes $\pi_{ii}, \pi_{ij}, \pi_{ik}$ be coaxial, the parameters K_i then having the values

$$K_i = -\frac{4}{5}(A\epsilon_i^2 + B\epsilon_i + 2) \quad (i = 1, 2, 3).$$

The three quadrics thus found have their residual point of intersection on the canonical line of the first kind for which $k = -9/20$.

The locus of the line of intersections of the planes π_{ij}, π_{ik} for all quadrics

in the pencil (19) is a plane passing through the Segre tangent $X_3/X_2 = \epsilon_i$, $X_4 = 0$ and through the canonical line of the first kind for which $k = -3/4$. The quadric determined by the condition that the planes π_{ij} , π_{ik} contain this canonical line corresponds to the parametric value

$$K_i = -2A\epsilon_i^2 - 2B\epsilon_i - 4,$$

and the three quadrics so obtained for $i = 1, 2, 3$ have their residual point of intersection on this same line.

5. The Moutard pencils of quadrics. This section is concerned essentially with an examination of the singularity of the curve of intersection of the surface S and a quadric of the Moutard pencil. The curve determined by the Moutard quadric of the pencil is found to differ radically from the curves determined by the other quadrics of the pencil.

The Moutard pencil of quadrics determined at a point 0 of S by a direction $X_3/X_2 = \mu$ has the form

$$(20) \quad XY - Z + (1/\mu)(2 - \mu^3)XZ + (1/\mu^2)(2\mu^3 - 1)YZ + K_\mu Z^2 = 0,$$

where K_μ is an arbitrary function of u, v . The quadric of Moutard of the pencil is obtained when

$$(21) \quad \mu^3 K_\mu = A\mu^2 + B\mu^4 - (\mu^3 + 1)^2.$$

Let the equation

$$(22) \quad Z = n(Y - \mu X) \quad (n \neq 0)$$

represent an arbitrary plane through the tangent $X_3/X_2 = \mu$, $X_4 = 0$, cutting the surface S in a curve s and a quadric of the pencil (20) in a conic q . Then s and q have third-order contact at 0 for all values of n and K_μ . Conversely, if every plane of the form (22) meets the surface S and a quadric Q in a pair of curves which have contact of the third order at 0, then Q belongs to the pencil (20). The quadric of Moutard (9) is the unique quadric of the pencil for which the contact between s and q is of the fourth order for all values of n .

If μ is not a Darboux direction, there are two planes of sextactic section⁸ passing through the tangent $X_3/X_2 = \mu$, $X_4 = 0$, for which the contact between s and q is of the fifth order at 0. The values of n in (22) which determine these planes are the roots of the equation

$$(23) \quad hn^2 + (3 - 3\mu^6 - A\mu^2 + B\mu^4)\mu n + (\mu^3 + 1)\mu^2 = 0,$$

⁸ Darboux, *Bulletin des Sciences Mathématiques et Astronomiques*, ser. 2, vol. 4 (1880), p. 366.

where h is defined by

$$h = 2(\mu^3 + 1)^3 - 3\mu^2(\mu^3 + 1)(A + B\mu^2) + \mu^2\alpha,$$

and

$$(24) \quad \alpha = \sum_0^5 C_i \mu^i.$$

The curve of intersection of S and an arbitrary quadric of the pencil (20) is given by equations (20) and (11). If μ is not a Darboux direction, this curve has two branches at 0 in the direction μ , whose equations are

$$(25) \quad Y = \mu X \pm rX^{3/2} + \dots, \quad Z = XY + \dots,$$

where r satisfies the equation

$$r^2 = -\mu(A\mu^2 + B\mu^4 - \mu^6 - 2\mu^3 - 1 - \mu^3 K_\mu)/(\mu^3 + 1).$$

If $r = 0$ the curve (25) is found to have a tacnode in the direction μ , so that in view of (21) *a necessary and sufficient condition that a quadric of the Moutard pencil in a non-Darboux direction μ at 0 be the quadric of Moutard (9) is that its curve of intersection with S have a tacnode at 0 in the direction μ instead of a cusp.*

For the case of the Moutard quadric (9) the terms of fifth degree in the series (11) can be shown to be

$$\sum_0^5 T_i X^{5-i} Y^i$$

where

$$\begin{aligned} T_0 &= C_0, & \mu^3 T_1 &= \mu^3 C_1 + \mu^9 - 6\mu^6 + 12\mu^3 - 8, \\ \mu^4 T_2 &= \mu^4 C_2 - 9\mu^9 + 3B\mu^7 + 27\mu^6 + 3A\mu^5 - 6B\mu^4 - 27\mu^3 - 6A\mu^2 + 18, \\ \mu^5 T_3 &= \mu^5 C_3 + 18\mu^9 - 6B\mu^7 - 27\mu^6 - 6A\mu^5 + 3B\mu^4 + 27\mu^3 + 3A\mu^2 - 9, \\ \mu^6 T_4 &= \mu^6 C_4 - 8\mu^9 + 12\mu^6 - 6\mu^3 + 1, & T_5 &= C_5. \end{aligned}$$

The two tacnode branches of the curve of intersection of the surface S and the Moutard quadric (9) are now found to have the equations

$$(26) \quad Y = \mu X + \sigma_i X^2 + \dots, \quad Z = XY + \dots \quad (i = 1, 2),$$

where σ_1, σ_2 are the roots of the equation

$$(27) \quad (\mu^3 + 1)\sigma^2 + (3 - 3\mu^6 - A\mu^2 + B\mu^4)\sigma + h = 0.$$

If $r \neq 0$, the osculating plane at 0 of each cusp-branch (25) coincides with the tangent plane $Z = 0$, whereas if $r = 0$ the osculating planes at 0 are represented by the equations

$$(28) \quad Z = (\mu/\sigma_i)(Y - \mu X) \quad (i = 1, 2).$$

Setting $\mu/\sigma_i = n_i$ and using (27) we find that n_1, n_2 are the roots of equation (23). Hence, the osculating planes at 0 of the tacnode branches of the curve of intersection of the surface S and the Moutard quadric in the non-Darboux direction μ at 0 coincide with the planes of sextactic section which pass through the tangent line in the direction μ .

Comparison of the discriminants of (23) and (27) shows that the two planes of sextactic section through the line $X_3/X_2 = \mu, X_4 = 0$ coincide if, and only if, the tacnode branches (26) have maximum-order contact at 0, this situation occurring for twelve values⁹ of μ .

The results concerning the Moutard pencil of quadrics in a Darboux direction $X_3/X_2 = -\epsilon_i$ at 0 will be stated briefly. Each quadric Q of the pencil meets the surface S in a curve C whose triple-point tangents all coincide in the given Darboux direction. If Q is not the Moutard quadric of the pencil, this curve has a superlinear branch of order 3 and the osculating plane at 0 of each branch coincides with the plane $Z = 0$. If Q is the Moutard quadric then the curve C consists of two cusped branches and one linear branch. The osculating planes of the cusped branches at 0 coincide with the tangent plane to S , while the osculating plane of the linear branch at 0 is none other than the plane (15). This plane can also be obtained as a limiting case of (28) as μ approaches $-\epsilon_i$.

6. Cones associated with the Moutard pencils of quadrics. The problem of characterizing individual members of the Moutard pencils of quadrics in the Segre directions at a point of a surface has already been discussed. We now consider this problem for an arbitrary direction.

The curve of intersection of the surface S and an arbitrary quadric of the Moutard pencil (20) in a non-Darboux direction μ at 0 has a linear branch in the direction $-1/\mu^2$. The equations of this branch are

$$Y = -\frac{1}{\mu^2}X + \frac{A\mu^8 + B\mu^4 - (\mu^6 - 1)^2 + \mu^6 K_\mu}{\mu^6(\mu^3 + 1)^2}X^2 + \cdots, \quad Z = XY + \cdots,$$

so that its osculating plane at 0 is of the form

$$(29) \quad \mu^2(\mu^3 + 1)(X + \mu^2 Y) + [A\mu^8 + B\mu^4 - (\mu^6 - 1)^2 + \mu^6 K_\mu]Z = 0.$$

The condition that this plane contain a line l joining 0 to an arbitrary

⁹ Wilczynski, "Fifth memoir," *Transactions of the American Mathematical Society*, vol. 10 (1909), p. 293.

point $(0, a, b, 1)$ determines a unique quadric Q_μ of the pencil, since the parameter K_μ must satisfy the equation

$$(30) \quad \mu^6 K_\mu = (\mu^6 - 1)^2 - A\mu^8 - a(\mu^3 + 1)^2 \mu^2 - B\mu^4 - b(\mu^3 + 1)^2 \mu^4.$$

Associated with the direction μ are two others, $\mu\omega$, $\mu\omega^2$ which form with μ a triple of directions apolar to the asymptotic directions. Thus two other quadrics, $Q_{\mu\omega}$, $Q_{\mu\omega^2}$, belonging to the Moutard pencils at 0 in the directions $\mu\omega$, $\mu\omega^2$, can be defined by this line l in exactly the same way as is Q_μ , the equations of these quadrics being obtained from (20) and (30) by replacing μ by $\mu\omega$, $\mu\omega^2$, respectively. The three quadrics Q_μ , $Q_{\mu\omega}$, $Q_{\mu\omega^2}$ intersect in the asymptotic tangents through 0 and in a residual point whose join with 0 is a line l' containing the point

$$(31) \quad \begin{aligned} X_1 &= 0, & X_2 &= (2\mu^3 - 1)[A\mu^6 + a(\mu^3 + 1)^2], \\ X_3 &= \mu^3(2 - \mu^3)[B + b(\mu^3 + 1)^2], & X_4 &= \mu^3(2 - \mu^3)(2\mu^3 - 1). \end{aligned}$$

The lines l and l' coincide if, and only if,

$$(32) \quad a = -A\mu^6/(2\mu^6 + 1), \quad b = -B/(\mu^6 + 2),$$

so that the line l and the quadric Q_μ are uniquely determined by the given direction μ . As μ varies, l generates the quadric cone given by

$$(33) \quad 3XY + 2AYZ + 2B\lambda Z + ABZ^2 = 0.$$

If, however, the line l is kept fixed and the direction μ is varied, then l' generates a cone which in general is of the fourth order but for special positions of l may be of the second order. To determine the conditions under which this occurs, we note that any quadric cone generated by the line l' must be of the form

$$(34) \quad XY + pYZ + qXZ + rZ^2 = 0,$$

where p , q , r are functions of u , v . Substitution of equations (31) into (34) leads to seven equations of which the following five are independent:

$$\begin{aligned} b(p - a - A) &= 0, & a(q - b - B) &= 0, \\ Ab - 3ab - 5pb - 4q(a + A) + 4r &= 0, \\ aB - 3ab - 5aq - 4p(b + B) + 4r &= 0, \\ 5AB + Ab + aB + 14ab + p(11b - 9B) + (11a - 9A)q + 33r &= 0. \end{aligned}$$

These equations have five sets of solutions for a , b , p , q , r which are exhibited in the following table:

Case	a	b	p	q	r
(I)	$-A/3$	$-B/3$	$2A/3$	$2B/3$	$AB/3$
(II)	0	0	$-A/3$	$-B/3$	$-AB/3$
(III)	$-4A/9$	$-4B/9$	$5A/9$	$5B/9$	$7AB/27$
(IV)	0	$-4B/9$	A	B	$5AB/9$
(V)	$4A/9$	0	A	B	$5AB/9$

Hence, a necessary and sufficient condition that the line l' generate a quadric cone as μ varies is that the fixed line l , which is the axis of the pencil of planes (29), join 0 to the point $(0, a, b, 1)$ where a, b have any one of the five pairs of values listed in the table.

Five quadrics belonging to the Moutard pencil in an arbitrary non-Darboux direction $^{10} \mu$ at 0 are thus characterized, each quadric having associated with it a quadric cone and a fixed line l . In cases (I) and (II) of the table this line is the first axis of Bompiani and the first edge of Green respectively, while in case (III) it is the canonical line of the first kind for which $k = -17/36$. The lines determined in cases (IV) and (V) can be described as the polars with respect to a quadric of Darboux of the lines $A_X B_Y, A_Y B_X$, where A_X, A_Y and B_X, B_Y are the intersections of the asymptotic tangents through 0 with the canonical lines of the second kind for which $k = -17/36, -1/4$.

The quadric cones associated with these five quadrics are all mutually tangent along the first axis of Bompiani, the common tangent plane passing through the second canonical tangent. Consideration of these cones readily leads to the characterization of a number of new canonical lines of the first kind.

The cone (33), which is also one of the cones listed in the table, can be obtained in still another way. The line joining 0 to the residual point of intersection of the three quadrics of Moutard in the directions $\mu, \mu\omega, \mu\omega^2$ generates the cone (33) as μ varies.¹¹

There is a certain cone of class 3 closely associated with the Moutard quadrics at 0. The quadrics of Moutard in the conjugate directions $\mu, -\mu$ meet in the asymptotic tangents through 0 and in a conic whose plane has the equation

$$2\mu^2(X + \mu^2 Y) + (A\mu^2 + B\mu^4 - \mu^6 - 1)Z = 0.$$

As μ varies this plane envelopes the cone, which in local plane coördinates $\eta_1, \dots, \eta_4$ has the equations

¹⁰ If μ is a Darboux direction these quadrics, as well as the quadric determined by (32), all coincide with the Moutard quadric.

¹¹ Bompiani, "Contributi alla geometria proiettivo-differenziale di una superficie," *Bollettino della Matematica Italiana*, vol. 3 (1924), pp. 97-100.

$$\eta_2^3 + \eta_3^3 - \eta_2\eta_3(A\eta_2 + B\eta_3 - 2\eta_4) = 0 = \eta_1.$$

The cusp-planes of this cone pass through the Segre tangents at 0 and intersect in the first directrix of Wilczynski.

7. Envelopes of Moutard quadrics. The problem of determining the envelope of a family of Moutard quadrics is of sufficient importance to warrant consideration. This problem has been discussed only by Kimpara,¹² to the best of the writer's knowledge, and then merely for a special case.

The one-parameter family of Moutard quadrics at a point 0 of a surface S has an enveloping surface Σ which contains the asymptotic tangents through 0 as a locus of singular points. Elimination of μ between (9) and its μ -derivative shows that Σ is an algebraic surface of order 14. The characteristic curve determined by Σ and a particular quadric of the family in the direction μ is a conic lying in the osculating plane at 0 of the pangeodesic which passes through 0 in the direction μ . Hence, *the lines joining 0 to the points on the edge of regression of Σ generate the cone of Segre.*

We now proceed with the problem of determining the envelope of a two-parameter family of Moutard quadrics. Let $C^{(1)}$ be a one-parameter family of curves on S which satisfy the differential equation

$$(35) \quad dv - \lambda du = 0 \quad (\lambda = \lambda(u, v)),$$

and let it be assumed that these curves simply cover the portion of the surface under consideration and are nowhere tangent to an asymptotic curve. The tangents to $C^{(1)}$ then determine a two-parameter family of Moutard quadrics $Q^{(2)}$, one at each point of S . A particular point 0 therefore has associated with it a curve C_μ of the family $C^{(1)}$, whose tangent at 0 is

$$X_3/X_2 = \mu, \quad X_4 = 0 \quad (\mu = \lambda l/m),$$

and a quadric Q_μ of the family $Q^{(2)}$. An arbitrary curve C_ν on S , which passes through 0 in a direction $X_3/X_2 = \nu$, determines at each of its points a unique Moutard quadric belonging to $Q^{(2)}$, the quadrics so obtained forming a one parameter family $Q^{(1)}$ with an envelope E_ν . The quadric Q_μ at 0 is a member of this family and hence touches E_ν along a characteristic curve Γ_ν .

These curves Γ_ν and the envelope E of the family of quadrics $Q^{(2)}$ depend on the given curves $C^{(1)}$, and their properties differ widely in two separate cases, namely, when the curves $C^{(1)}$ are curves of Darboux and when they are nowhere tangent to a curve of Darboux.

¹² "Sur l'enveloppe des quadriques de Moutard," *Ryojun College of Engineering, Commemoration volume* (1934), pp. 33-39.

(i) Suppose first that the curves $C^{(1)}$ form a one-parameter family of curves of Darboux. The quadric Q_μ is the Moutard quadric in the Darboux direction $X_3/X_2 = \mu$ at 0, and has the equation

$$(36) \quad G \equiv X_2 X_3 - X_1 X_4 - 3\mu^2 X_2 X_4 + 3\mu X_3 X_4 + (B\mu - A\mu^2) X_4^2 = 0.$$

The characteristic curve Γ_v determined by Q_μ and a curve C_v through 0 is therefore given by (36) and

$$G_u + G_v(dv/du) = 0,$$

where dv/du refers to the direction of the tangent to C_v at 0, and since $dv/du = vm/l$ this may be written

$$(37) \quad lG_u + vmG_v = 0.$$

The derivatives of A and B which appear in this equation may be readily evaluated. For,

$$\begin{aligned} A &= -(3/2)m\psi, & A_u &= -(3/2)(m_u\psi + m\psi_u), \\ m &= 1/(\beta\gamma^2)^{1/3}, & m_u &= -(m/3)(\beta_u/\beta) - (2m/3)(\gamma_u/\gamma), \end{aligned}$$

and consequently,

$$18lA_u = 4AB - 27m\psi_u.$$

Now from (3) it is seen that $n\psi_u$ can be expressed in terms of the coefficients appearing in series (2), and hence lA_u is expressible in like fashion. Thus we find

$$\begin{aligned} 36lA_u &= 278AB - 648 - 48C_1 - 24C_4, \\ 36mB_v &= 278AB - 648 - 48C_4 - 24C_1, \\ 36mA_v &= 52A^2 + 288B - 24C_2 - 120C_5, \\ 36lB_u &= 52B^2 + 288A - 24C_3 - 120C_0. \end{aligned}$$

With the aid of formulas (4) the left member of (37) can now be expressed in terms of μ , v , the coefficients of series (2), and the local coördinates $X_1, \dots, X_4$. Elimination of the term $X_1 X_4$ from the resulting equation by means of (36) finally yields the simplified result

$$(38) \quad 36(\mu - v)(\mu X_2 - X_3) + LX_2 X_4 + MX_3 X_4 + NX_4^2 = 0,$$

where the coefficients L , M , N have the values

$$\begin{aligned} L &= 12B\mu(\mu + 2v) - 36A + 216(v - \mu), \\ M\mu &= 12\mu(A\mu + B)(\mu - v) - L, \\ N &= 72A(2v - \mu) - 72B\mu^2(2\mu - v) + B^2(54v + 40\mu) - A^2\mu^2(54\mu + 40v) \\ &\quad + 131AB\mu(v - \mu) + 12 \sum_0^5 C_i \mu^i [(5 - i)\mu + iv]. \end{aligned}$$

If $\nu \neq \mu$, the equations of Γ_ν can also be written in the non-homogeneous form

$$Y = \mu X \pm \frac{1}{6} \left(\frac{L\mu + M\mu^2}{\nu - \mu} \right)^{1/2} X^{3/2} + \dots, \quad Z = XY + \dots \quad (\nu \neq \mu),$$

which simplify to

$$Y = \mu X \pm \left[\frac{1}{3} (A - B\mu^2) \right]^{1/2} X^{3/2} + \dots, \quad Z = XY + \dots \quad (\nu \neq \mu).$$

Hence, if $\nu \neq \mu$, the characteristic curve Γ_ν is a quartic curve with a cusp at 0 and a cusp-tangent in the Darboux direction μ .

If C_ν is tangent at 0 to the curve of Darboux C_μ , then the characteristic curve Γ_ν consists of the asymptotic tangents through 0 and of a non-degenerate conic lying in the plane whose equation is

$$(39) \quad 18(A\mu + B)(\mu X_2 - X_3) \\ + [30\alpha - 36\mu^2(A\mu + B) - 47(A^2\mu^2 - B^2)]X_4 = 0,$$

α being defined in (24). Since this conic is tangent to C_μ at 0, it follows that if C_ν and C_μ coincide, then C_μ is part of the edge of regression of the envelope E_ν .

The points where the Quadric Q_μ touches the envelope E of the two-parameter family of Moutard quadrics $Q^{(2)}$ are determined by the simultaneous solutions of equations (36), (38) and (39), where in (38) the value of ν is arbitrary but different from μ . Since the plane (39) intersects the cone (38) in the Darboux tangent $X_3/X_2 = \mu$, $X_4 = 0$ and in another line l which passes through 0, and since these two lines meet the quadric Q_μ at 0 and at some other point P not lying on S , it therefore follows that *the envelope E consists of the surface S and of another sheet generated by the point P .*

(ii) We now assume that the curves $C^{(1)}$ are nowhere tangent to a curve of Darboux. If $H = 0$ is the equation (9) of the Moutard quadric Q_μ in the direction μ at 0, then the characteristic curve Γ_ν determined by Q_μ and a curve C_ν through 0 has the form

$$H = 0, \quad lH_u + \nu mH_v = 0.$$

The second of these equations can be expressed in terms of μ , ν , the coefficients of series (2), and the local coördinates $X_1, \dots, X_4$, and when combined with (9) to eliminate the term X_1X_4 becomes

$$(40) \quad 12\mu^4(\mu X_2 - X_3) [(2\nu - 3\mu - \mu^3\nu)X_2 - (2\mu^3 - 3\mu^3\nu - 1)X_3] \\ + DX_2X_4 + EX_3X_4 + FX_4^2 = 0,$$

where D, E, F have the values

$$\begin{aligned}
 D &= 4A\mu^5(2\mu^3\nu - 9\mu + 2\nu) + 4B\mu^5(\mu^3 + 6\mu^2\nu - 2) \\
 &\quad + 12\mu^3[\mu^6(\mu - 4\nu) - 10\mu^3(\mu - \nu) + 7\mu - 4\nu] \\
 &\quad - 72\mu^4(\mu^3 + 1)(l\mu_u + m\mu_v), \\
 E\mu &= 12\mu^3(\nu - \mu)(3\mu^6 - 3 + A\mu^2 - B\mu^4) - D, \\
 F &= 12A\mu^4[(5\mu^3 + 11) + \mu^2\nu(\mu^3 + 13)] \\
 &\quad + 12B\mu^3[(13\mu^3 + 1) + \mu^2\nu(11\mu^3 + 5)] \\
 &\quad + A^2\mu^5(54\mu + 40\nu) + B^2\mu^6(40\mu + 54\nu) \\
 &\quad + AB\mu^4[\mu(45\mu^3 + 176) + \nu(176\mu^3 + 45)] \\
 &\quad + 12\mu[\mu(\mu^9 - 18\mu^6 - 21\mu^3 - 2) + \nu(-2\mu^9 - 21\mu^6 - 18\mu^3 + 1)] \\
 &\quad - 12\mu^3 \sum_0^5 C_i \mu^i [(5 - i)\mu + i\nu] \\
 &\quad + 36\mu^2(3 - 3\mu^6 - A\mu^2 + B\mu^4)(l\mu_u + m\mu_v).
 \end{aligned}$$

The characteristic curve Γ_ν is therefore a quartic curve with a double point at 0, one nodal tangent always lying in the given direction μ , the other depending on ν . This second tangent lies in the direction μ if, and only if, $\nu = \mu$, and lies in the direction ν only if $\nu = \mu$ or $\nu = -1/\mu^2$. The osculating plane at 0 of that branch of Γ_ν which lies in the direction μ has the equation

$$24\mu^4(\mu^3 + 1)(\mu - \nu)(\mu X_2 - X_3) - (D + E\mu)X_4 = 0,$$

which reduces, when $\nu \neq \mu$, to (8). Hence, if $\nu \neq \mu$, the osculating plane at 0 of that nodal branch of the characteristic curve Γ_ν which lies in the direction μ is independent of ν and is the osculating plane of the pangeodesic in the direction μ .

If $\nu \neq \mu$, the osculating plane of the second nodal branch of Γ_ν at 0 is expressible in the form

$$\begin{aligned}
 24\mu^4(\mu^3 + 1)(\mu - \nu)[(2\nu - 3\mu - \mu^3\nu)X_2 - (2\mu^3 - 3\mu^2\nu - 1)X_3] \\
 + [D(2\mu^3 - 3\mu^2\nu - 1) + E(2\nu - 3\mu - \mu^3\nu)]X_4 = 0.
 \end{aligned}$$

As ν varies this plane generates an axial pencil, its limiting position as ν approaches μ being given by

$$\begin{aligned}
 (41) \quad 6\mu(\mu^3 + 1)(\mu X_2 - X_3) \\
 + [A\mu^2(11 - 4\mu^3) + B\mu(4 - 11\mu^3) + 9\mu^6 - 9 + 36(\mu^3 + 1)\mu']X_4 = 0,
 \end{aligned}$$

where μ' is defined in (7). The plane (41) depends on the curve C_μ alone, being independent of the other curves of the family $C^{(1)}$, and together with the osculating plane of C_μ at 0, the osculating plane of the pangeodesic in the direction μ at 0, and the tangent plane $X_4 = 0$, determines an invariant cross-

ratio of -4 . Hence, the plane (41) coincides at every point of the surface S with the osculating plane of the curve C_μ if, and only if, the curves $C^{(1)}$ are pangeodesics.

If $\nu = \mu$, (40) reduces to

$$(42) \quad [X_4 - n_1(X_3 - \mu X_2)][X_4 - n_2(X_3 - \mu X_2)] = 0,$$

where n_1, n_2 are the roots of the equation

$$(43) \quad Fn^2 + En - 12\mu^4(\mu^3 + 1) = 0 \quad (\nu = \mu).$$

The characteristic curve Γ_ν therefore consists of two non-degenerate conics determined by the intersections of the quadric Q_μ and the planes (42). The harmonic conjugate of the plane $X_4 = 0$ with respect to these two planes has the form

$$24\mu^4(\mu^3 + 1)(\mu X_2 - X_3) + EX_4 = 0 \quad (\nu = \mu),$$

from which it follows that this plane, the osculating plane of C_μ at 0, the osculating plane of the pangeodesic in the direction μ at 0, and the plane $X_4 = 0$, determine a cross-ratio of -2 .

The curves $C^{(1)}$ cannot be chosen so that at each point of S the two planes (42) coincide with the planes of sextactic section (28) which pass through the tangents to $C^{(1)}$. However, comparison of equations (43) and (23) shows that this situation does occur at a particular point 0 of S if, and only if, C_μ has second-order contact at 0 with any one of a set of twelve pangeodesics.

The quadric Q_μ touches the envelope E of the two-parameter family of Moutard quadrics $Q^{(2)}$ at the points determined by equations (9), (40) and (42), where the value of ν in (40) is arbitrary but different from μ . Since the two planes (42) meet the cone (40) in two lines l_1, l_2 which pass through 0 and are ordinarily distinct, and in the tangent $X_3/X_2 = \mu, X_4 = 0$, and since these three lines intersect Q_μ at 0 and at two other points P_1, P_2 , it follows that the envelope E consists of the surface S and of two other sheets, those generated by the points P_1, P_2 .

The lines l_1, l_2 coincide if, and only if,

$$(44) \quad E^2 + 48F\mu^4(\mu^3 + 1) = 0,$$

where $\nu = \mu$ in the expressions for E, F . Hence, at a particular point 0 of S there are two planes either of which may serve as the osculating plane of the curve C_μ in order that l_1 and l_2 coincide. It is impossible, however, to choose the curves $C^{(1)}$ so that the lines l_1, l_2 coincide for every point of S , since equation (35) which determines the curves $C^{(1)}$ cannot satisfy (44) which is now a differential equation of the second order and second degree in dv/du .

